

Matrix Rank

Example 1: Show that the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is row equivalent to each of the following matrices in row echelon form.

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad (1)$$

$$\begin{aligned}
 A &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{\substack{R_2 := R_2 - 7R_1 \\ R_3 := R_3 - 4R_1}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \xrightarrow{R_3 := R_3 - 2R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} = B \xrightarrow{R_2 := -\frac{1}{3}R_2} \\
 &\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = C \xrightarrow{R_1 := R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = D
 \end{aligned}$$

Theorem 1: If a matrix A is row equivalent to matrices B and C both in row echelon form, then the number of nonzero rows of B is equal to the number of nonzero rows of C .

Definition: The rank of a matrix A , denoted rank(A), is the number of nonzero rows in any matrix B in row echelon form that is row equivalent to A .

Example 1 Continued: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ then

$$\text{rank}(A) = \underline{2}$$

Example 2: Calculate the rank of the following matrix.

$$A = \begin{bmatrix} 1 & -2 & 1 & 4 \\ -1 & 2 & 1 & 2 \\ 2 & -4 & 0 & 2 \end{bmatrix} \quad (2)$$

$$A = \begin{bmatrix} \textcircled{1} & -2 & 1 & 4 \\ -1 & 2 & 1 & 2 \\ 2 & -4 & 0 & 2 \end{bmatrix} \quad \begin{array}{l} R_2 := R_2 - 2R_1 \\ R_3 := R_3 + R_1 \end{array} \quad \begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 0 & \textcircled{2} & 6 \\ 0 & 0 & -3 & -6 \end{bmatrix} \quad R_3 := R_3 + R_2 \quad \left. \begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\} \text{Non zero rows}$$

$$\text{rank}(A) = 2$$